Amplitude equations
for a linear wave equation in a weakly curved pipe

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October 3, 2009

Abstract

We study boundary effects in a linear wave equation with Dirichlet type conditions in a weakly curved pipe. The coordinates in our pipe are prescribed by a given small curvature with finite range, while the pipe's cross section being circular. Based on the straight pipe case a perturbative analysis by which the boundary value conditions are exactly satisfied is employed. As such an analysis we decompose the wave equation into a set of ordinary differential equations perturbatively. We show the conditions when secular terms due to the curbed boundary appear in the naive peturbative analysis. In eliminating such a secularity with a singular perturbation method, we derive amplitude equations and show that the eigenfrequencies in time are shifted due to the curved boundary.

1 Introduction

In a spatially extended system a perfunctory theoretical description to explain experimentally observed phenomena could be done by ignoring or simplifying boundary conditions. However in reality, the observed phenomena in finite domain are generally affected by spatial boundary conditions and are non-trivial. A reason why boundary conditions are ignored or simplified is the lack of mathematical tools. It has been difficult to study such boundary effects in a systematic manner. We then need to develop a theoretical framework that enables us to describe some effects of spatial boundary conditions. A first attempt might rely on a perturbative analysis, and the small parameter is related to the magnitude of deformation from trivial spatial boundary. In dynamical systems theory there are a variety of useful methodologies for dealing with perturbed systems[1]. With those methodologies differential geometry provides powerful mathematical tools for it[2]. Beyond some existing works on this context, such as electromagnetic waves in a curved pipe[3, 4], quantum eigenstates of a curved nanowire[5], one would like to know the higher order corrections in the magnitude of deformation of boundaries. To clearly see what can be observed including higher orders due to non-trivial spatial boundary conditions one needs to have a simple model. As such a model, we consider a linear wave equation which is widely studied in physical sciences, for example, fluid dynamics, electromagnetism and high energy physics. In this paper the linear wave equation, or strictly speaking the classical complex Klein-Gordon equation, with Dirichlet type conditions is studied based on the language of differential geometry with the use of

perturbation methods. There we will derive amplitude equations. In fact the idea of amplitude equations is often used in order to study weakly perturbed systems in nonlinear science and give benefits. We will see that even in the linear equation a mode-coupling phenomenon occurs due to the weakly deformed boundary, and then perturbative correction terms for the mode-amplitude will be obtained using a singular perturbation method.

2   Coordinate system, co-frame and Laplacians

First we give the expression for the coordinate system adapted to our curved pipe. The z-coordinate for our curved pipe which we consider is assumed to be planer and prescribed by the curvature which is small. Accordingly we do not consider the spatial curve, and our curve is nearly straight. This assumption on curvature makes the perturbative analysis effective. The two-dimensional cross section at any z in our pipe is assumed to be circle whose radius is a (See Fig.1).

We denote the given curvature by

$$\kappa(z) = \epsilon \kappa_0(z), \tag{1}$$

where $|\kappa(z)a| \ll 1$ and ϵ is the small parameter.

To describe the geometry of boundary we use the following metric tensor

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3,$$

where $\{e^1, e^2, e^3\}$ is the co-frame to be specified later, and our space will be denoted by $\mathcal{U}$. The canonical volume elements are defined as

$$\#1 = e^1 \wedge e^2 \wedge e^3, \qquad \widehat{\#}1 = e^1 \wedge e^2.$$

Then the Hodge maps $\#$, $\widehat{\#}$ are defined :

$$\# : \Gamma\Lambda^q \mathcal{U} \to \Gamma\Lambda^{3-q} \mathcal{U}, \qquad \widehat{\#} : \Gamma\Lambda^q \mathcal{U} \to \Gamma\Lambda^{2-q} \mathcal{U},$$

acting on $\{e^1, e^2, e^3\}$ and $\{e^1, e^2\}$, respectively. Here $\Gamma\Lambda^q \mathcal{U}$ denotes the set of q-form fields on $\mathcal{U}$.

For our weakly curved pipe the adapted co-frame is derived from the use of Frenet frame. The space-curve is given in terms of Euclidean position vector $\boldsymbol{C}(z)$ then the points in the interior can be written as

$$\boldsymbol{r}(z, r, \theta) = \boldsymbol{C}(z) + x_1(r, \theta)\, \boldsymbol{n}(z) + x_2(r, \theta)\, \boldsymbol{b}(z).$$
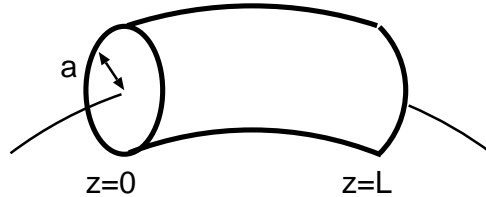

Figure 1: Curved pipe

Here x_1 and x_2 can be chosen as $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ with $0 \le r \le a$ and $0 \le \theta < 2\pi$. In addition, $\boldsymbol{n}$ is the Frenet normal and $\boldsymbol{b}$ is the Frenet bi-normal vectors to the curve, the tangent vector $\boldsymbol{t}$ is given by $d\boldsymbol{C}/dz$. The relations between them are describes as the Frenet-Serret formulae:

$$\frac{d}{dz}\boldsymbol{C} = \boldsymbol{t}, \qquad \frac{d}{dz}\boldsymbol{t} = \kappa\boldsymbol{n}, \qquad \frac{d}{dz}\boldsymbol{n} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \qquad \frac{d}{dz}\boldsymbol{b} = -\tau\boldsymbol{n},$$

with $\tau(z)$ being the Frenet torsion, which is zero in our case. Since the infinitesimal deviation of $\boldsymbol{r}$ is written with $\delta x_1, \delta x_2$ and δz that are infinitesimal deviations of x_1, x_2 and z,

$$\delta\boldsymbol{r} = \boldsymbol{n}(\delta x_1 - x_2\tau\delta z) + \boldsymbol{b}(\delta x_2 + x_1\tau\delta z) + \boldsymbol{t}(1 - \kappa x_1)\delta z,$$

one can choose a convenient orthonormal co-frame with $\tau = 0$ for the interior domain

$$\{ \quad e^1 = dr \quad e^2 = rd\theta, \quad e^3 = (1 - \epsilon\kappa_0(z)r\cos\theta)dz \quad \},$$

Taking the limit $\epsilon \to 0$ above one has the adapted co-frame for the straight pipe

$$\{ \quad e^1 = dr \quad e^2 = rd\theta, \quad e^3 = dz \quad \}.$$

These are the usual cylindrical ones.

2.1 Laplacians

The Laplacian on $\mathcal{U}$ for differentiable functions is

$$\# d \# d, \tag{2}$$

where $d : \Gamma\Lambda^p\mathcal{U} \to \Gamma\Lambda^{p+1}\mathcal{U}$ is the exterior differentiation in a patch with coordinates in $\mathcal{U}$. So, for any function of t, z, r, θ one has

$$df(t, z, r, \theta) = \frac{\partial f}{\partial r}dr + \frac{\partial f}{r\partial\theta}d\theta + \frac{\partial f}{\partial z}dz.$$

The explicit form of (2) for functions is calculated to be

$$\# d \# d = \left\{ \frac{1}{(1 - \kappa(z)r\cos\theta)^2}\frac{\partial^2}{\partial z^2} + \frac{\kappa'(z)r\cos\theta}{(1 - \kappa(z)r\cos\theta)^3}\frac{\partial}{\partial z} \right\}$$
$$+ \left\{ \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - \frac{\kappa(z)\cos\theta}{1 - \kappa(z)r\cos\theta}\frac{\partial}{\partial r} \right\} + \left\{ \frac{\partial^2}{r^2\partial\theta^2} + \frac{\kappa(z)\sin\theta}{r(1 - \kappa(z)r\cos\theta)}\frac{\partial}{\partial\theta} \right\},$$

where $'$ denotes the differentiation with respect to z. When $\kappa = 0$ this expression of the Laplacian reduces to

$$\triangle^{(0)} := \# d \# d\Big|_{\epsilon=0} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}. \tag{3}$$

Higher order corrections are obtained by substituting $\kappa(z) = \epsilon\kappa_0(z)$, (1), and expanding it in ϵ as

$$\# d \# d = \triangle^{(0)} + \epsilon\triangle^{(1)} + \epsilon^2\triangle^{(2)} + \cdots,$$

3

where

$$\triangle^{(1)} := 2\kappa_0(z)r\cos\theta\frac{\partial^2}{\partial z^2} + \kappa_0'(z)r\cos\theta\frac{\partial}{\partial z} - \kappa_0(z)\cos\theta\frac{\partial}{\partial r} + \kappa_0(z)\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\triangle^{(2)} := 3(\kappa_0(z)r\cos\theta)^2\frac{\partial^2}{\partial z^2} + 3\kappa_0(z)\kappa_0'(z)(r\cos\theta)^2\frac{\partial}{\partial z} - (\kappa_0(z)\cos\theta)^2 r\frac{\partial}{\partial r} + \kappa_0^2(z)\sin\theta\cos\theta\frac{\partial}{\partial\theta}.$$

Similarly the two-dimensional Laplacian for any function of (r,θ) is generalized to

$$\widehat{\#}\,d\,\widehat{\#}\,d,$$

The perturbation due to the curvature does not affect the two-dimensional Laplacian, and then the explicit form of this is calculated as

$$\widehat{\#}\,d\,\widehat{\#}\,d = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}.$$

2.2 Eigenvalue problems of Laplacians at $\epsilon = 0$

The eigenvalue problem associated with the two-dimensional Laplacian, acting on functions of (r,θ) that satisfy the Dirichlet condition $\Phi_N(a,\theta) = 0$, is written as

$$\widehat{\#}\,d\,\widehat{\#}\,d\,\Phi_N(r,\theta) = -\beta_N^2\Phi_N(r,\theta).$$

An explicit form of the solution Φ_N is found to be

$$\Phi_N(r,\theta) = J_n\left(\frac{x_{q(n)}}{a}r\right)e^{in\theta},$$

and the constant β_N is

$$\beta_N = \frac{x_{q(n)}}{a}, \qquad J_n(x_{q(n)}) = 0,$$

with $N = \{n, q(n)\}, n = 0, \pm 1, \pm 2, \ldots$, and $q = 1, 2, \ldots$, J_n is the n-th order Bessel function and $x_{q(n)}$ is the q-th zero of J_n. The orthogonality can be shown as

$$\int_{\mathcal{D}} \overline{\Phi_M}\Phi_N\widehat{\#}1 = \mathcal{N}_N^2\delta_{N,M}. \tag{4}$$

where

$$\mathcal{D} := \{\ (r,\theta)\ \ |\ \ 0 \leq r \leq a, \quad 0 \leq \theta < 2\pi\ \ \},$$
$$\mathcal{N}_N^2 = \pi a^2 J_{n+1}^2(x_{q(n)}), \qquad \delta_{N,M} := \delta_{n,m}\delta_{q,p}$$

with $N = \{n, q(n)\}$, $M = \{m, p(m)\}$ and $\delta_{a,b}$ being the Kronecker delta symbol.

The eigenvalue problem associated with the three-dimensional Laplacian at $\epsilon = 0$, acting on functions of (z, r, θ) that satisfy Dirichlet condition $\varphi_{N,\eta} = 0$ at boundary, is solved as

$$\triangle^{(0)}\varphi_{N,\eta}(z,r,\theta) = -\left\{\beta_N^2 + \left(\frac{\eta\pi}{L}\right)^2\right\}\varphi_{N,\eta}(z,r,\theta), \qquad \varphi_{N,\eta}(z,r,\theta) = \Phi_N(r,\theta)\sin\left(\frac{\eta\pi}{L}z\right).$$

Here $\eta = 1, 2, \ldots$ and the boundary is located at

$$z = 0, L, \qquad r = a.$$

The eigenfunction $\varphi_{N,\eta}$ also satisfies

$$\left(\triangle^{(0)} + \mu^2\right)\varphi_{N,\eta}(z,r,\theta) = -\left\{\beta_N^2 + \left(\frac{\eta\pi}{L}\right)^2 + \mu^2\right\}\varphi_{N,\eta}(z,r,\theta), \tag{5}$$

with the same boundary conditions where μ is a constant.

4

3 The wave equation in the curved pipe and its naive perturbative analysis

We study the following complex-valued linear partial differential equation for $\phi(\epsilon, t, z, r, \theta) \in \mathbb{C}$

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \# d \# d + \mu^2 \right\} \phi(\epsilon, t, z, r, \theta) = 0, \tag{6}$$

where c, μ are constants, the operators appeared here are defined in §2, and the ranges of coordinates are

$$-\infty \leq t \leq \infty, \qquad 0 \leq z \leq L, \qquad 0 \leq r \leq a, \qquad 0 \leq \theta < 2\pi.$$

The boundary condition is

$$\phi(\epsilon, t, z, r, \theta)|_{\partial \mathcal{U}} = 0, \tag{7}$$

where $\partial \mathcal{U}$ denotes spatial boundary. In this paper (6) is referred to as the wave equation.

To obtain an approximate solution we expand the solution based on the lowest order problem

$$\phi(\epsilon, t, z, r, \theta) = \sum_{\eta=1}^{\infty} \sum_{N} \Phi_N(r, \theta) \sin\left(\frac{\eta\pi}{L} z\right) A_{N,\eta}(\epsilon, t),$$

$$A_{N,\eta}(\epsilon, t) = A_{N,\eta}^{(0)}(t) + \epsilon A_{N,\eta}^{(1)}(t) + \epsilon^2 A_{N,\eta}^{(2)}(t) + \mathcal{O}(\epsilon^3), \tag{8}$$

and correspondingly

$$\begin{aligned} \phi(\epsilon, t, z, r, \theta) &= \phi^{(0)}(t, z, r, \theta) + \epsilon \phi^{(1)}(t, z, r, \theta) + \epsilon^2 \phi^{(2)}(t, z, r, \theta) + \mathcal{O}(\epsilon^3), \\ \phi^{(j)}(t, z, r, \theta) &:= \sum_{\check{N}} \Phi_N(r, \theta) \sin\left(\frac{\eta\pi}{L} z\right) A_{\check{N}}^{(j)}(t), \qquad j = 0, 1, 2, \dots, \end{aligned}$$

where $\check{N} := \{N, \eta\}$ which we call a mode. This expansion of the solution in ϵ and $A_{\check{N}}$ are refereed to as the naive perturbation expansion and the mode-amplitude associated with the mode $\check{N}$ respectively. Note here that this form of the expansion of the solution satisfies the boundary conditions, (7), for any value of ϵ.

To obtain equations for $A_{\check{N}}^{(j)}$ from those for $\phi^{(j)}$, one uses the following identity

$$A_{\check{N}}^{(j)}(t) = \frac{1}{\mathcal{N}_{\check{N}}^2} \int_{\mathcal{D}} \overline{\Phi_N} \widehat{\#} 1 \frac{2}{L} \int_0^L dz \, \sin\left(\frac{\eta\pi}{L} z\right) \phi^{(j)}(t, z, r, \theta). \tag{9}$$

3.1 Unperturbed solution

To zeroth order in ϵ the equation of motion becomes

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \triangle^{(0)} + \mu^2 \right\} \phi^{(0)}(t, z, r, \theta) = 0, \tag{10}$$

where $\triangle^{(0)}$ is defined in (3). This can be written in terms of $A_{\check{N}}^{(0)}$ using (5) and (9) as

$$\ddot{A}_{\check{N}}^{(0)} + \Omega_{\check{N}}^2 A_{\check{N}}^{(0)} = 0, \qquad \Omega_{\check{N}} := c \sqrt{\left(\frac{\eta\pi}{L}\right)^2 + \beta_N^2 + \mu^2},$$

where $\dot{\,}$ denotes the differentiation with respect to t. The solution is

$$A_{\check{N}}^{(0)}(t) = A_{\check{N}}^{(0,-)} e^{-i\Omega_{\check{N}} t} + A_{\check{N}}^{(0,+)} e^{i\Omega_{\check{N}} t}, \tag{11}$$

where $A_{\check{N}}^{(0,\pm)} \in \mathbb{C}$ are integral constants.

3.2 First order solution

To first order in ϵ the equation of motion is written as

$$\left\{ \frac{1}{c^2} \frac{\partial}{\partial t^2} - \triangle^{(0)} + \mu^2 \right\} \phi^{(1)}$$

$$= 2\kappa_0(z)\phi^{(0)\prime\prime} r \cos\theta + \kappa_0'(z)\phi^{(0)\prime\prime} r \cos\theta - \kappa_0(z)\frac{\partial\phi^{(0)}}{\partial r} \cos\theta + \kappa_0(z)\frac{\partial\phi^{(0)}}{\partial\theta} \frac{\sin\theta}{r},$$

or equivalently,

$$\ddot{A}_{\check{N}}^{(1)} + \Omega_{\check{N}}^2 A_{\check{N}}^{(1)} = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\check{N}'} F_{\check{N},\check{N}'}^{(1)} A_{\check{N}'}^{(0)}(t). \tag{12}$$

Here

$$F_{\check{N},\check{N}'}^{(1)} := \left\{ -2\left(\frac{\eta'\pi}{L}\right)^2 \langle\kappa_0\rangle_{S_\eta,S_{\eta'}} + \frac{\eta'\pi}{L}\langle\kappa_0'\rangle_{S_\eta,C_{\eta'}} \right\} C_{N,N'}^{(1,1)} + \langle\kappa_0\rangle_{S_\eta,S_{\eta'}} C_{N,N'}^{(1,2)}, \quad \in \mathbb{R}$$

$$C_{N,M}^{(1,1)} := \int_{\mathcal{D}} \overline{\Phi_N} \Phi_M r \cos\theta \, \widehat{\#}1 = \pi(\delta_{m,n-1} + \delta_{m,n+1}) \int_0^a dr r^2 J_n\left(\frac{x_{q(n)}}{a}r\right) J_m\left(\frac{x_{p(m)}}{a}r\right), \quad \in \mathbb{R}$$

$$C_{N,M}^{(1,2)} := \int_{\mathcal{D}} \overline{\Phi_N}\left(\frac{\partial\Phi_M}{\partial\theta}\frac{\sin\theta}{r} - \frac{\partial\Phi_M}{\partial r}\cos\theta\right)\widehat{\#}1$$

$$= \pi m(\delta_{m,n-1} - \delta_{m,n+1}) \int_0^a dr J_n\left(\frac{x_{q(n)}}{a}r\right) J_m\left(\frac{x_{p(m)}}{a}r\right)$$

$$\quad -\pi\frac{x_{p(m)}}{a}(\delta_{m,n-1} + \delta_{m,n+1}) \int_0^a dr r J_n\left(\frac{x_{q(n)}}{a}r\right) J_m'\left(\frac{x_{p(m)}}{a}r\right). \quad \in \mathbb{R}$$

In addition, for a function f of z, we have defined

$$\langle f \rangle_{S_\eta,S_{\eta'}} := \int_0^L dz \sin\left(\frac{\eta\pi}{L}z\right)\sin\left(\frac{\eta'\pi}{L}z\right) f(z), \quad \langle f \rangle_{S_\eta,C_{\eta'}} := \int_0^L dz \sin\left(\frac{\eta\pi}{L}z\right)\cos\left(\frac{\eta'\pi}{L}z\right) f(z).$$

Due to $\delta_{n',n\pm 1}$ in $C_{N,N'}^{(1,1)}$ and $C_{N,N'}^{(1,2)}$ the right hand side of (12) reduces to

$$\frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\check{N}'} F_{\check{N},\check{N}'}^{(1)} A_{\check{N}'}^{(0)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm 1} \sum_{q'(n')} F_{\check{N},n'q'(n'),\eta'}^{(1)} A_{n',q'(n'),\eta'}^{(0)}(t).$$

After substituting (11) into (12) one can find the solution (See §6)

$$A_{\check{N}}^{(1)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm 1} \sum_{q'(n')} \left\{ R_{\check{N},\check{N}'} F_{\check{N},n',q'(n'),\eta'}^{(1)} t \left(\frac{A_{\check{N}'}^{(0,-)}}{-2i\Omega_{\check{N}}} e^{-i\Omega_{\check{N}} t} + \frac{A_{\check{N}'}^{(0,+)}}{2i\Omega_{\check{N}}} e^{i\Omega_{\check{N}} t} \right) \right.$$

$$\left. + N_{\check{N},\check{N}'} F_{\check{N},n',q'(n'),\eta'}^{(1)} \frac{A_{\check{N}'}^{(0,-)} e^{-i\Omega_{\check{N}'} t} + A_{\check{N}'}^{(0,+)} e^{i\Omega_{\check{N}'} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}'}^2} \right\}, \tag{13}$$

where $R_{\check{N},\check{N}'}$ and $N_{\check{N},\check{N}'}$ are defined as

$$R_{\check{N},\check{N}'} := \delta_{\Omega_{\check{N}},\Omega_{\check{N}'}}, \qquad N_{\check{N},\check{N}'} := 1 - \delta_{\Omega_{\check{N}},\Omega_{\check{N}'}}.$$

The condition $R_{\check{N},\check{N}'} = 1$ at this order is equivalent to the resonance conditions

$$\Omega_{n,q(n),\eta} = \Omega_{n\pm1,q'(n\pm1),\eta'}. \tag{14}$$

On the other hand the condition $N_{\check{N},\check{N}'} = 1$ at this order is the case when

$$\Omega_{n,q(n),\eta} \neq \Omega_{n\pm1,q'(n\pm1),\eta'}.$$

Note that there are secular terms, $\propto t$, in (13) when the resonance condition is satisfied, and that the resonance condition, (14), does not contain $\kappa(z)$.

3.3 Second order solution

To second order in ϵ the equation of motion is written as

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \triangle^{(0)} + \mu^2 \right\} \phi^{(2)}$$

$$= 2\kappa_0(z)\phi^{(1)''}r\cos\theta + \kappa_0'(z)\phi^{(1)'}r\cos\theta - \kappa_0(z)\frac{\partial\phi^{(1)}}{\partial r}\cos\theta + \kappa_0(z)\frac{\partial\phi^{(1)}}{\partial\theta}\frac{\sin\theta}{r}$$

$$+3(\kappa_0(z)r\cos\theta)^2\phi^{(0)''} + 3\kappa_0(z)\kappa_0'(z)(r\cos\theta)^2\phi^{(0)'} - \kappa_0^2(z)r\frac{\partial\phi^{(0)}}{\partial r}\cos^2\theta + \kappa_0^2(z)\frac{\partial\phi^{(0)}}{\partial\theta}\cos\theta\sin\theta,$$

or equivalently,

$$\ddot{A}_{\check{N}}^{(2)} + \Omega_{\check{N}}^2 A_{\check{N}}^{(2)} = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}\left\{ \sum_{n'=n\pm1}\sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} A_{\check{N}'}^{(1)}(t) + \sum_{n'=n,n\pm2}\sum_{q'(n')} F_{\check{N},\check{N}'}^{(2)} A_{\check{N}'}^{(0)}(t) \right\}. \tag{15}$$

Here

$$F_{\check{N},\check{N}'}^{(2)} := \left\{ -3\left(\frac{\eta'\pi}{L}\right)^2 \langle\kappa_0^2\rangle_{S_\eta,S_{\eta'}} + 3\frac{\eta'\pi}{L}\langle\kappa_0\kappa_0'\rangle_{S_\eta,C_{\eta'}} \right\} C_{N,N'}^{(2,1)} - \langle\kappa_0^2\rangle_{S_\eta,S_{\eta'}} C_{N,N'}^{(2,2)} + \frac{i}{2}n'\langle\kappa_0^2\rangle_{S_\eta,S_{\eta'}} C_{N,N'}^{(2,3)},$$

with

$$C_{N,N'}^{(2,1)} := \int_{\mathcal{D}} \overline{\Phi_N}\Phi_{N'}r^2\cos\theta\,\widehat{\#}1$$

$$= \pi\left\{ \delta_{n',n} + \frac{1}{2}(\delta_{n',n+2} + \delta_{n',n-2}) \right\} \int_0^a dr\, r^3 J_n\left(\frac{x_{q(n)}}{a}r\right) J_{n'}\left(\frac{x_{q'(n')}}{a}r\right), \quad \in \mathbb{R}$$

$$C_{N,N'}^{(2,2)} := \int_{\mathcal{D}} \overline{\Phi_N}\frac{\partial\Phi_{N'}}{\partial r}r\cos^2\theta\,\widehat{\#}1$$

$$= \pi\left\{ \delta_{n',n} + \frac{1}{2}(\delta_{n',n+2} + \delta_{n',n-2}) \right\} \frac{x_{q'(n')}}{a} \int_0^a dr\, r^2 J_n\left(\frac{x_{q(n)}}{a}r\right) J_{n'}'\left(\frac{x_{q'(n')}}{a}r\right), \quad \in \mathbb{R}$$

$$C_{N,N'}^{(2,3)} := \int_{\mathcal{D}} \overline{\Phi_N}\Phi_{N'}\sin 2\theta\,\widehat{\#}1$$

$$= -\pi i(\delta_{n',n-2} - \delta_{n',n+2}) \int_0^a dr\, r J_n\left(\frac{x_{q(n)}}{a}\right) J_{n'}\left(\frac{x_{q'(n')}}{a}\right). \quad \in i\mathbb{R}$$

One obtains the equation of motion at this order after substituting the lower solutions $A_{\check{N}}^{(0)}(t)$ and $A_{\check{N}}^{(1)}(t)$, given in (11) and (13), into (15). The right hand side of (15) can then be expressed as

$$\frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \left[\frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \sum_{\eta''} \sum_{n''=n'\pm1} \sum_{q''(n'')} \right.$$

$$\times \left\{ R_{\check{N}',\check{N}''} F_{\check{N}',\check{N}''}^{(1)} t \left(\frac{A_{\check{N}''}^{(0,-)}}{-2i\Omega_{\check{N}'}} e^{-i\Omega_{\check{N}'} t} + \frac{A_{\check{N}''}^{(0,+)}}{2i\Omega_{\check{N}'}} e^{i\Omega_{\check{N}'} t} \right) \right.$$

$$+ N_{\check{N}',\check{N}''} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}''}^2} \left(A_{\check{N}''}^{(0,-)} e^{-i\Omega_{\check{N}''} t} + A_{\check{N}''}^{(0,+)} e^{i\Omega_{\check{N}''} t} \right) \bigg\} \bigg]$$

$$+ \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''} \sum_{n''=n,n\pm2} \sum_{q''(n'')} F_{\check{N},\check{N}''}^{(2)} \left(A_{\check{N}''}^{(0,-)} e^{-i\Omega_{\check{N},\check{N}''} t} + A_{\check{N}''}^{(0,+)} e^{i\Omega_{\check{N},\check{N}''} t} \right),$$

from which one has the solution

$$A_{\check{N}}^{(2)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \sum_{\eta''} \sum_{n''=n'\pm1} \sum_{q''(n'')}$$

$$\times \left[R_{\check{N}',\check{N}''} \frac{F_{\check{N}',\check{N}''}^{(1)}}{-2i\Omega_{\check{N}'}} A_{\check{N}''}^{(0,-)} \left\{ R_{\check{N},\check{N}''} \left(\frac{it^2}{4\Omega_{\check{N}}} + \frac{t}{4\Omega_{\check{N}}^2} \right) e^{-i\Omega_{\check{N}} t} + N_{\check{N},\check{N}''} t \frac{e^{-i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\} \right.$$

$$+ N_{\check{N}',\check{N}''} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}''}^2} A_{\check{N}''}^{(0,-)} \left\{ R_{\check{N},\check{N}''} \frac{t\, e^{-i\Omega_{\check{N}} t}}{-2i\Omega_{\check{N}}} + N_{\check{N},\check{N}''} \frac{e^{-i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\}$$

$$+ R_{\check{N}',\check{N}''} \frac{F_{\check{N}',\check{N}''}^{(1)}}{2i\Omega_{\check{N}'}} A_{\check{N}''}^{(0,+)} \left\{ R_{\check{N},\check{N}''} \left(\frac{-it^2}{4\Omega_{\check{N}}} + \frac{t}{4\Omega_{\check{N}}^2} \right) e^{i\Omega_{\check{N}} t} + N_{\check{N},\check{N}''} t \frac{e^{i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\}$$

$$+ N_{\check{N}',\check{N}''} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}''}^2} A_{\check{N}''}^{(0,+)} \left\{ R_{\check{N},\check{N}''} \frac{t\, e^{i\Omega_{\check{N}} t}}{2i\Omega_{\check{N}}} + N_{\check{N},\check{N}''} \frac{e^{i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\} \bigg]$$

$$+ \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''} \sum_{n''=n,n\pm2} \sum_{q''(n'')} F_{\check{N},\check{N}''}^{(2)} \left[A_{\check{N}''}^{(0,-)} \left\{ R_{\check{N},\check{N}''} \frac{t\, e^{-i\Omega_{\check{N}} t}}{-2i\Omega_{\check{N}}} + N_{\check{N},\check{N}''} \frac{e^{-i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\} \right.$$

$$+ A_{\check{N}''}^{(0,+)} \left\{ R_{\check{N},\check{N}''} \frac{t\, e^{i\Omega_{\check{N}} t}}{2i\Omega_{\check{N}}} + N_{\check{N},\check{N}''} \frac{e^{i\Omega_{\check{N}''} t}}{\Omega_{\check{N}}^2 - \Omega_{\check{N}''}^2} \right\} \bigg]. \tag{16}$$

When the resonance condition at $\mathcal{O}(\epsilon)$ is satisfied, the solution at $\mathcal{O}(\epsilon^2)$ includes the terms being proportional to t^2. From (16) the resonance conditions between different two modes are obtained as

$$\Omega_{n,q(n),\eta} = \Omega_{n\pm2,q''(n\pm2),\eta''}. \tag{17}$$

Although when (17) are not satisfied in addition to the case where (14) are not satisfied, there are secular terms in $A_{\check{N}}^{(2)}(t)$ due to the self-mode couplings. The existence of such secular behavior in $A_{\check{N}}^{(2)}(t)$ is a notable qualitative difference from the analysis at $\mathcal{O}(\epsilon)$. Such secular behavior is due

to the summation ranges in (16)

$$\sum_{n''=n,n\pm1}, \quad \text{and} \quad \sum_{n'=n\pm1}\sum_{n''=n'\pm1} = \sum_{n'=n\pm1}\sum_{n''=n,n\pm2}.$$

Taking into account the resonance conditions up to $\mathcal{O}(\epsilon^2)$ one rewrites (16) explicitly in the following cases

(i)

$$\Omega_{n,q(n),\eta} \neq \Omega_{n\pm1,q'(n\pm1),\eta'}, \quad \text{for all} \quad q'(n\pm1), \eta' \quad \text{and}$$
$$\Omega_{n,q(n),\eta} \neq \Omega_{n\pm1,q''(n\pm2),\eta''}, \quad \text{for all} \quad q''(n\pm2), \eta'',$$

(ii)

$$\Omega_{n,q(n),\eta} \neq \Omega_{n\pm1,q'(n\pm1),\eta'}, \quad \text{for all} \quad q'(n\pm1), \eta', \quad \text{and}$$
$$\Omega_{n,q(n),\eta} = \Omega_{n\pm2,q''_r(n\pm2),\eta''_r}, \quad \text{for some particular} \quad q''_r(n\pm2), \eta''_r, \quad (18)$$

(iii)

$$\Omega_{n,q(n),\eta} = \Omega_{n\pm1,q'_r(n\pm1),\eta'_r}, \quad \text{for some particular} \quad q'_r(n\pm1), \eta'_r, \quad \text{and} \quad (19)$$
$$\Omega_{n,q(n),\eta} = \Omega_{n\pm2,q''_r(n\pm2),\eta''_r}, \quad \text{for some particular} \quad q''_r(n\pm2), \eta''_r, \quad (20)$$

Here the reason why the case

$$\Omega_{n,q(n),\eta} = \Omega_{n\pm1,q'_r(n\pm1),\eta'_r}, \quad \text{for some particular} \quad q'_r(n\pm1), \eta'_r, \quad \text{and} \quad (21)$$
$$\Omega_{n,q(n),\eta} \neq \Omega_{n,q''(n\pm2),\eta''}, \quad \text{for all} \quad q''(n\pm2), \eta'', \quad (22)$$

does not appear is as follows. If (21) is satisfied, then one uses (21) twice, and can find $q''_r(n\pm2), \eta''_r$ that satisfy

$$\Omega_{n,q(n),\eta} = \Omega_{n,q''_r(n\pm2),\eta''_r}.$$

This is in contradiction to (22).

3.3.1   Case (i)

In the case, where both (14) and (17) are not satisfied, one has

$$A_{\check{N}}^{(2)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \left[t \left(\frac{A_{\check{N}}^{(0,-)}}{-2i\Omega_{\check{N}}} e^{-i\Omega_{\check{N}}t} + \frac{A_{\check{N}}^{(0,+)}}{2i\Omega_{\check{N}}} e^{i\Omega_{\check{N}}t} \right) \right.$$

$$\left. \times \left\{ \sum_{\eta'} \sum_{n'=n\pm1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \frac{F_{\check{N}',\check{N}}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}}^2} + F_{\check{N},\check{N}}^{(2)} \right\} \right] + (\,\text{NS}\,), \quad (23)$$

where $F_{\check{N},\check{N}}^{(2)}$ is real because $C_{n,q(n),n,q'(n)}^{(2,3)} \in i\,\mathbb{R}$ vanishes and NS is the abbreviation for the nonresonance terms. The time-dependence of NS is $\exp(i\,\Omega'_{\check{N}'}t)$ with $\Omega'_{\check{N}'} \neq \Omega_{\check{N}}$ for a given $\check{N}$.

Thus when the resonance conditions between other different modes are not satisfied one observes a secular (divergent in time) behavior, $A_{\check{N}}^{(2)}(t) \propto t$. Correspondingly, our naive perturbative analysis could only be valid in a short time range. To improve this naive perturbative result one needs another perturbation method that will be discussed in §4.

9

3.3.2 Case (ii)

The solution in the case (ii) is written as

$$A_{\check{N}}^{(2)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm 1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2$$

$$\times \sum_{\eta''}^{(\mathrm{res})} \sum_{n''=n'\pm 1}^{(\mathrm{res})} \sum_{q''(n'')}^{(\mathrm{res})} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}}^2} t \left(\frac{A_{\check{N}}^{(0,-)}}{-2i\Omega_{\check{N}}} e^{-i\Omega_{\check{N}}t} + \frac{A_{\check{N}}^{(0,+)}}{2i\Omega_{\check{N}}} e^{i\Omega_{\check{N}}t} \right)$$

$$+ \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''}^{(\mathrm{res})} \sum_{n''=n'\pm 1}^{(\mathrm{res})} \sum_{q''(n'')}^{(\mathrm{res})} F_{\check{N},\check{N}''}^{(2)} t \left(\frac{A_{\check{N}''}^{(0,-)}}{-2i\Omega_{\check{N}}} e^{-i\Omega_{\check{N}}t} + \frac{A_{\check{N}''}^{(0,+)}}{2i\Omega_{\check{N}}} e^{i\Omega_{\check{N}}t} \right) + (\,\mathrm{NS}\,), \quad (24)$$

where

$$\sum_{\eta''}^{(\mathrm{res})} \sum_{n''=n'\pm 1}^{(\mathrm{res})} \sum_{q''(n'')}^{(\mathrm{res})}$$

are the summations of η'', n'' and q'' that satisfy (18) and $\Omega_{n,q(n),\eta} = \Omega_{n,q''(n),\eta''}$, the self-mode coupling, for a given set $\check{N}$.

3.3.3 Case (iii)

The solution in the case (iii) is written as

$$A_{\check{N}}^{(2)}(t) = \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(\mathrm{res})} \sum_{n'=n\pm 1}^{(\mathrm{res})} \sum_{q'(n')}^{(\mathrm{res})} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2$$

$$\times \sum_{\eta''}^{(\mathrm{res})} \sum_{n''=n'\pm 1}^{(\mathrm{res})} \sum_{q''(n'')}^{(\mathrm{res})} F_{\check{N}',\check{N}''}^{(1)} \left\{ \frac{A_{\check{N}''}^{(0,-)}}{-2i\Omega_{\check{N}'}} \left(\frac{it^2}{4\Omega_{\check{N}}} + \frac{t}{4\Omega_{\check{N}}^2} \right) e^{-i\Omega_{\check{N}}t} + \frac{A_{\check{N}''}^{(0,+)}}{2i\Omega_{\check{N}'}} \left(\frac{-it^2}{4\Omega_{\check{N}}} + \frac{t}{4\Omega_{\check{N}}^2} \right) e^{i\Omega_{\check{N}}t} \right\}$$

$$+ \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''}^{(\mathrm{res})} \sum_{n''=n'\pm 1}^{(\mathrm{res})} \sum_{q''(n'')}^{(\mathrm{res})} F_{\check{N},\check{N}''}^{(2)} t \left(\frac{A_{\check{N}''}^{(0,-)}}{-2i\Omega_{\check{N}}} e^{-i\Omega_{\check{N}}t} + \frac{A_{\check{N}''}^{(0,+)}}{2i\Omega_{\check{N}}} e^{i\Omega_{\check{N}}t} \right) + (\,\mathrm{NS}\,), \quad (25)$$

where the summations are performed when (19), (20) and $\Omega_{n,q(n),\eta} = \Omega_{n,q''(n),\eta''}$, the self-mode coupling, are satisfied for a given set $\check{N}$.

4 Amplitude equations

In this section, we derive amplitude equations by which long time behavior can approximately be described for each case in §3.3.

The procedure to obtain an amplitude equation will be shown in the case (i), where there is no resonance between different modes. Even in this case as mentioned in §3.3.1, there is a secular term, $\propto \epsilon^2 t$ due to the self-mode coupling. This does not make our naive perturbative result valid globally in time, and valid only for a short time range. To improve this one uses another perturbation method, rather than the naive perturbation method. In this paper we use the

10

renormalization method as a type of systematic perturbation method for removing secular terms caused by the use of the naive perturbation method. Using the renormalization method, one can obtain the equations of motion that describe the long-time behavior[6],[7],[8],[9],[10]. In this paper we use the method in Ref.[10] by which reduced equations have been systematically obtained[11].

For the other cases, (ii) and (iii) in §3.3, the procedures to obtain amplitude equations are similar to the case (i), and we then show only the resultant equations.

4.1 case (i)

In the case (i) one concentrates on (23), which contains secular terms due to the use of the naive perturbation expansion.

First, to collect secular terms in (23) perturbatively one defines the following polynomials

$$A_{\check{N}}^{(pol,-)}(t) \quad := \quad A_{\check{N}}^{(0,-)} - i\epsilon^2 t\,\Omega_{\check{N}}^{(ren,\,2)}\,A_{\check{N}}^{(0,-)}, \tag{26}$$

$$A_{\check{N}}^{(pol,+)}(t) \quad := \quad A_{\check{N}}^{(0,+)} + i\epsilon^2 t\,\Omega_{\check{N}}^{(ren,\,2)}\,A_{\check{N}}^{(0,+)}, \tag{27}$$

where

$$\Omega_{\check{N}}^{(ren,\,2)} := -\frac{1}{L\Omega_{\check{N}}}\left(\frac{c}{\mathcal{N}_N}\right)^2\left[\sum_{\eta'}\sum_{n'=n\pm 1}\sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)}\frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2\frac{F_{\check{N}',\check{N}}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}}^2} + F_{\check{N},\check{N}}^{(2)}\right]. \quad \in \mathbb{R}$$

From these definitions, the naive solution can approximately be written as

$$A_{\check{N}}(\epsilon,t) \quad = \quad A_{\check{N}}^{(0)}(t) + \epsilon A_{\check{N}}^{(1)}(t) + \epsilon^2 A_{\check{N}}^{(2)}(t) + \mathcal{O}(\epsilon^3)$$

$$= \quad A_{\check{N}}^{(pol,-)}(t)\,e^{-i\Omega_{\check{N}}t} + A_{\check{N}}^{(pol,+)}(t)\,e^{i\Omega_{\check{N}}t} + \mathcal{O}(\epsilon^3) + (\,\mathrm{NS}\,). \tag{28}$$

Second, one derives the equations which (26) and (27) should perturbatively satisfy. From the definitions one has

$$\frac{A_{\check{N}}^{(pol,\mp)}(t+\tau) - A_{\check{N}}^{(pol,\mp)}(t)}{\tau} = \mp i\epsilon^2\Omega_{\check{N}}^{(ren,\,2)}A_{\check{N}}^{(0,\mp)},$$

where τ is a real constant. The right hand sides contain $A_{\check{N}}^{(0,\mp)}$. To obtain the closed equations in terms of $A_{\check{N}}^{(pol,\mp)}(t)$, not $A_{\check{N}}^{(0,\mp)}$, one substitutes $A_{\check{N}}^{(0,\mp)} = A_{\check{N}}^{(pol,\mp)}(t) + \mathcal{O}(\epsilon^2)$ which are the inverse of the definitions (26) and (27), then

$$\frac{A_{\check{N}}^{(pol,\mp)}(t+\tau) - A_{\check{N}}^{(pol,\mp)}(t)}{\tau} = \mp i\epsilon^2\Omega_{\check{N}}^{(ren,\,2)}A_{\check{N}}^{(pol,\mp)}(t) + \mathcal{O}(\epsilon^4).$$

Taking the limit, $\tau \to 0$, for the both sides one has the renormalization equations

$$\frac{d}{dt}A_{\check{N}}^{(ren,\mp)} \quad = \quad \mp i\epsilon^2\Omega_{\check{N}}^{(ren,\,2)}A_{\check{N}}^{(ren,\mp)} \tag{29}$$

where $A_{\check{N}}^{(ren,\mp)}(t)$ approximate $A_{\check{N}}^{(pol,\mp)}(t)$, and the solutions to (29) contain higher order terms in ϵ. The explicit forms of the solutions are

$$A_{\check{N}}^{(ren,\mp)}(t) = A_{\check{N}}^{(ren,\mp)}(0)e^{\mp i\epsilon^2\Omega_{\check{N}}^{(ren,\,2)}t}. \tag{30}$$

11

Finally one obtains an approximate solution for $A_{\check{N}}(\epsilon, t)$ that does not contain the secular terms. Using (28) and (30) one has

$$A_{\check{N}}(\epsilon, t) \approx A_{\check{N}}^{(ren,-)}(0)\, e^{-i\Omega_{\check{N}}^{(ren)}t} + A_{\check{N}}^{(ren,+)}(0)\, e^{i\Omega_{\check{N}}^{(ren)}t}.$$

The frequency-shift due to the curved boundary is obtained as

$$\Omega_{\check{N}}^{(ren)} := \Omega_{\check{N}} + \epsilon^2 \Omega_{\check{N}}^{(ren,\,2)} + \mathcal{O}(\epsilon^3).$$

4.2   case (ii)

We give the renormalization equations for the case (ii). The procedure to obtain these equations is same as in §4.1.

To collect the secular terms one defines

$$
\begin{aligned}
A_{\check{N}}^{(pol,\mp)}(t) \quad := \quad & A_{\check{N}}^{(0,\mp)} + \epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm 1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \\
& \times \overset{(\mathrm{res})}{\sum_{\eta''}} \overset{(\mathrm{res})}{\sum_{n''=n'\pm 1}} \overset{(\mathrm{res})}{\sum_{q''(n'')}} \frac{t}{\mp 2i\Omega_{\check{N}}} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}}^2} A_{\check{N}}^{(0,\mp)} \\
& + \epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \overset{(\mathrm{res})}{\sum_{\eta''}} \overset{(\mathrm{res})}{\sum_{n''=n'\pm 1}} \overset{(\mathrm{res})}{\sum_{q''(n'')}} \frac{t}{\mp 2i\Omega_{\check{N}}} F_{\check{N},\check{N}''}^{(2)} A_{\check{N}''}^{(0,\mp)}.
\end{aligned}
$$

From these equations one can obtain the renormalization equations

$$
\begin{aligned}
\frac{dA_{\check{N}}^{(ren,\mp)}}{dt} \quad = \quad & \epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'} \sum_{n'=n\pm 1} \sum_{q'(n')} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \\
& \times \overset{(\mathrm{res})}{\sum_{\eta''}} \overset{(\mathrm{res})}{\sum_{n''=n'\pm 1}} \overset{(\mathrm{res})}{\sum_{q''(n'')}} \frac{1}{\mp 2i\Omega_{\check{N}}} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\Omega_{\check{N}'}^2 - \Omega_{\check{N}}^2} A_{\check{N}}^{(ren,\mp)} \\
& + \epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \overset{(\mathrm{res})}{\sum_{\eta''}} \overset{(\mathrm{res})}{\sum_{n''=n'\pm 1}} \overset{(\mathrm{res})}{\sum_{q''(n'')}} \frac{1}{\mp 2i\Omega_{\check{N}}} F_{\check{N},\check{N}''}^{(2)} A_{\check{N}''}^{(ren,\mp)}.
\end{aligned}
\tag{31}
$$

Unlike the case (i), the right hand side of (31) consists of some different mode amplitudes, then the frequency-shift cannot directly be obtained from this form of the renormalization equations. One can rewrite (31) as

$$\frac{d}{dt} A_{\check{N}}^{(ren,\mp)} = \pm i\epsilon^2 \sum_{\check{N}'} M_{\check{N},\check{N}'}^{(ii)} A_{\check{N}'}^{(ren,\mp)}, \tag{32}$$

where $\{M_{\check{N},\check{N}'}^{(ii)}\}$ is a real matrix.

4.3 case (iii)

Finally, defining

$$
A_{\check{N}}^{(pol,\mp)}(t) := A_{\check{N}}^{(0,\mp)} + \epsilon \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(res)} \sum_{n'=n\pm1}^{(res)} \sum_{q'(n')}^{(res)} \frac{F_{\check{N},\check{N}'}^{(1)}}{\mp 2i\Omega_{\check{N}}} t A_{\check{N}'}^{(0,\mp)}
$$

$$
+\epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(res)} \sum_{n'=n\pm1}^{(res)} \sum_{q'(n')}^{(res)} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2
$$

$$
\times \sum_{\eta''}^{(res)} \sum_{n''=n'\pm1}^{(res)} \sum_{q''(n'')}^{(res)} \frac{F_{\check{N}',\check{N}''}^{(1)}}{-2i\Omega_{\check{N}'}}\left(\frac{\pm it^2}{4\Omega_{\check{N}}} + \frac{t}{4\Omega_{\check{N}}^2}\right) A_{\check{N}'}^{(0,\mp)}
$$

$$
+\epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''}^{(res)} \sum_{n''=n'\pm1}^{(res)} \sum_{q''(n'')}^{(res)} \frac{t}{\mp 2i\Omega_{\check{N}}} F_{\check{N},\check{N}''}^{(2)} A_{\check{N}''}^{(0,\mp)}, \tag{33}
$$

one has

$$
\frac{dA_{\check{N}}^{(ren,\mp)}}{dt} = \epsilon \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(res)} \sum_{n'=n\pm1}^{(res)} \sum_{q'(n')}^{(res)} \frac{F_{\check{N},\check{N}'}^{(1)}}{\mp 2i\Omega_{\check{N}}} A_{\check{N}'}^{(ren,\mp)}
$$

$$
+\epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(res)} \sum_{n'=n\pm1}^{(res)} \sum_{q'(n')}^{(res)} F_{\check{N},\check{N}'}^{(1)} \frac{2}{L}\left(\frac{c}{\mathcal{N}_{N'}}\right)^2 \sum_{\eta''}^{(res)} \sum_{n''=n'\pm1}^{(res)} \sum_{q''(n'')}^{(res)} \frac{F_{\check{N}',\check{N}''}^{(1)}}{\mp 2i\Omega_{\check{N}'}} \frac{1}{4\Omega_{\check{N}}^2} A_{\check{N}'}^{(ren,\mp)}
$$

$$
+\epsilon^2 \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta''}^{(res)} \sum_{n''=n'\pm1}^{(res)} \sum_{q''(n'')}^{(res)} \frac{1}{\mp 2i\Omega_{\check{N}}} F_{\check{N},\check{N}''}^{(2)} A_{\check{N}''}^{(ren,\mp)}. \tag{34}
$$

Here in the course of deriving the renormalization equations we have substituted

$$
A_{\check{N}}^{(0,\mp)} = A_{\check{N}}^{(pol,\mp)}(t) - \epsilon t \frac{2}{L}\left(\frac{c}{\mathcal{N}_N}\right)^2 \sum_{\eta'}^{(res)} \sum_{n'=n\pm1}^{(res)} \sum_{q'(n')}^{(res)} \frac{F_{\check{N},\check{N}'}^{(1)}}{\mp 2i\Omega_{\check{N}}} A_{\check{N}'}^{(pol)}(t) + \mathcal{O}(\epsilon^2),
$$

which are from the definitions of $A_{\check{N}}^{(pol,\mp)}(t)$, (33). One can rewrite (34) in the matrix form as

$$
\frac{d}{dt} A_{\check{N}}^{(ren,\mp)} = \pm i\epsilon \sum_{\check{N}'} M_{\check{N},\check{N}'}^{(iii)} A^{(ren,\mp)},
$$

where $\{M_{\check{N},\check{N}'}^{(iii)}\}$ is a real matrix, as same as the case of (ii).

5 Discussion and Conclusions

In this paper we have explored physical effects due to non-trivial spatial boundary conditions for the system of a linear wave equation. As a simple example, we have studied the system with Dirichlet type boundary conditions in a prescribed weakly curved pipe. The perturbation scheme

was based on the non-perturbed eigenvalue problems of the Laplacians and gave us the set of ordinary differential equations. Then it has been observed that self-mode couplings occur and then secular terms appear at the second order analysis, in addition to the secular terms due to resonances between different modes. The resonance conditions have been derived at each order in the naive perturbative analysis and it turns out that these resonance conditions do not contain $\kappa(z)$.Using the renormalization method we have derived amplitude equations. In the case where self-mode coupling only occurs we have obtained an analytical expression for the frequency-shift due to the curved boundary. In the case where there are resonances between different modes, we have derived the amplitude equations in a matrix form as renormalization equations. The amplitude equation with higher order correction terms can be obtained by applying our procedure straightforwardly in each case. Beyond this study, our methodology can be applied to systems with other boundary conditions, for example, the prescribed curve is not planer and the cross section is rectangular. In the case where the cross section is rectangular, Bessel functions as the eigenfunctions for the two-dimensional Laplacian will be replaced with sinusoidal functions. Furthermore, if the given system has a weakly nonlinear term, there will be nonlinear resonance terms in the naive perturbation expansion in addition to those associated with curved boundary. Thus it is obvious from our procedure to obtain the amplitude equations that the corresponding amplitude equations will be nonlinear. We believe that our present work and these extensions that follow from this work can help elucidate the behavior of systems with non-trivial spacial boundary conditions.

6 Appendix

In this appendix we study the following problems

$$\ddot{A} + \Omega^2 A = (\alpha_1^{(-)} t + \alpha_0^{(-)}) \, e^{-i\Omega t} + (\alpha_1^{(+)} t + \alpha_0^{(+)}) \, e^{i\Omega t}, \tag{35}$$

and

$$\ddot{A} + \Omega^2 A = (\beta_1^{(-)} t + \beta_0^{(-)}) \, e^{-i\Omega' t} + (\beta_1^{(+)} t + \beta_0^{(+)}) \, e^{i\Omega' t}, \tag{36}$$

Here $\dot{}$ denotes the differentiation with respect to t, $\alpha_j^{(\pm)}, \beta_j^{(\pm)} \in \mathbb{C}$ and $\Omega, \Omega'(\neq \pm\Omega) \in \mathbb{R}$ are given constants.

The solution to (35) is obtained as

$$A(t) = \left\{ \frac{\alpha_1^{(-)}}{-4i\Omega} t^2 + \left(\frac{\alpha_0^{(-)}}{-2i\Omega} + \frac{\alpha_1^{(0)}}{4\Omega^2} \right) t \right\} e^{-i\Omega t} + \left\{ \frac{\alpha_1^{(+)}}{4i\Omega} t^2 + \left(\frac{\alpha_0^{(+)}}{2i\Omega} + \frac{\alpha_1^{(0)}}{4\Omega^2} \right) t \right\} e^{i\Omega t}.$$

Similarly, the solution to (36) is obtained as

$$
\begin{aligned}
A(t) &= \left\{ \frac{\beta_1^{(-)}}{\Omega^2 - \Omega'^2} t + \frac{\beta_0^{(-)}}{\Omega^2 - \Omega'^2} + 2i\Omega' \frac{\beta_1^{(-)}}{(\Omega^2 - \Omega'^2)^2} \right\} e^{-i\Omega' t} \\
&\quad + \left\{ \frac{\beta_1^{(+)}}{\Omega^2 - \Omega'^2} t + \frac{\beta_0^{(-)}}{\Omega^2 - \Omega'^2} - 2i\Omega' \frac{\beta_1^{(+)}}{(\Omega^2 - \Omega'^2)^2} \right\} e^{i\Omega' t}.
\end{aligned}
$$

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